

# III Southern-Summer School on Mathematical Biology

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Lecture VI

São Paulo, February 2014



# Outline

## 1 Semi-arid and arid regions



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- 2 Model



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- 3 Hysteresis



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- 2 Model
- 3 Hysteresis
- 4 Glory and Misery of the Model



# Vegetation in semi-arid regions

Eremology: science of arid regions.

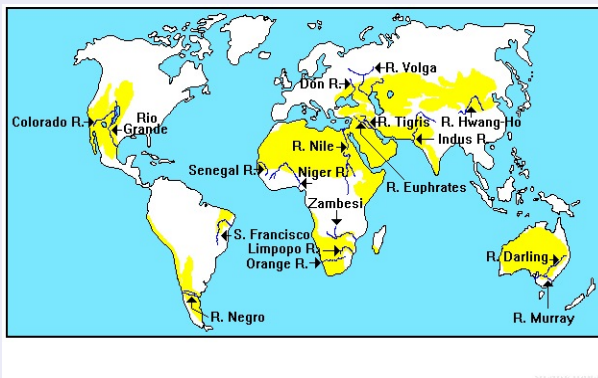


Figura : Arid and semi-arid regions of the world

# Vegetation in semi-arid regions

Eremology: science of arid regions.

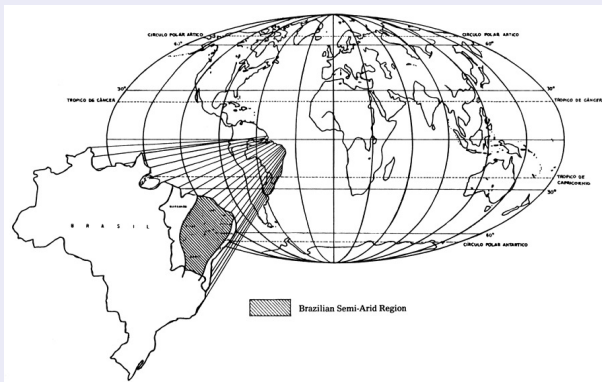


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# Vegetation in Semi-Arid Regions

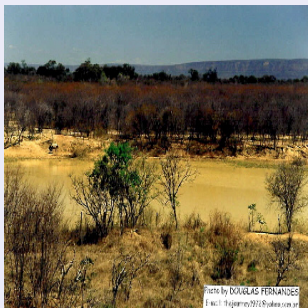


Figura : Bahia

- Consider the vegetation cover in water-poor regions.
- In this case, water is a *limiting factor*.
- quite different from tropical regions, where competition for water is irrelevant. One of the main limiting factor is light.
- We want to build a mathematical model — (**simple, please**) — to describe the mutual relation between water in soil and biomass in semi-arid regions.
- Let us do it



# Klausmeier Model



Figura : Colorado, USA



Figura : Kalahari, Namibia

- Water and vegetation, in a first approximation, entertain a relation similar to *predator-prey dynamics*.
- The presence of water is incremental for vegetation;
- Vegetation consumes water.
- But note that water does not originate from water.. It is an abiotic variable.
- The usual predator-prey dynamics does not apply.
- Consider two variables:
  - ▶  $w$ , the amount of water in soil.
  - ▶  $u$ , the vegetation biomass ( proportional to the area with vegetation cover).

# Klausmeier Model

## Equation for the amount of water in soil

$$\frac{dw}{dt} = \underbrace{a}_{\text{precipitation}} - \underbrace{bw}_{\text{evaporation}} - \underbrace{cu^2w}_{\text{absorption by vegetation}}$$

Water in soil increases due to precipitation ( $a$ ), evaporates at a constant *per volume* rate ( $b$ ), and is absorbed by vegetation in a per volume rate that depends on  $u^2$  ( $c$ ). This is phenomenological law coming from lab fittings



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## Equation for biomass

$$\frac{du}{dt} = \underbrace{-du}_{\text{natural death}} + \underbrace{eu^2w}_{\text{absorption of water}}$$

Vegetation has a natural death rate, ( $d$ ) and absorbs water at a per volume rate ( $e$ ) proportional to  $uw$ .

# Analysis

$$\frac{dw}{dt} = a - bw - cu^2w$$

$$\frac{du}{dt} = -du + eu^2w$$

- Let us begin by defining two new variables, rescaled ones:

$$W = w \left[ \frac{e}{\sqrt{b^3c}} \right]$$

$$U = u\sqrt{bc}$$

$$T = tb$$

- They are dimensionless.
- Plug them into the equations and you will get....



# Analysis of the model

$$\frac{dW}{dT} = A - W - WU^2$$

$$\frac{dU}{dT} = WU^2 - BU$$

where

$$A = \frac{ae}{\sqrt{b^3c}}$$

and

$$B = d/b$$

⇒ the equations depend only on **two** parameters, instead of five.

What do these equations tell us?

# Analysis of the model

$$\frac{dW}{dT} = A - W - WU^2$$

$$\frac{dU}{dT} = WU^2 - BU$$

- Let us look for fixed points:
- The points  $U^*$  e  $W^*$  such that



$$\frac{dW^*}{dT} = 0$$



$$\frac{dU^*}{dT} = 0$$

- or



$$A - W^* - W^*(U^*)^2 = 0$$



$$W^*(U^*)^2 - BU^* = 0$$

# Analysis of the model

$$\frac{dW}{dT} = A - W - WU^2$$

$$\frac{dU}{dT} = WU^2 - BU$$

- The algebraic equations has three roots:

$$U^* = 0$$

$$W^* = A$$

If  $A > 2B$

$$U^* = \frac{2B}{A - \sqrt{A^2 - 4B^2}}$$

$$W^* = \frac{A - \sqrt{A^2 - 4B^2}}{2}$$

If  $A > 2B$

$$U^* = \frac{2B}{A + \sqrt{A^2 - 4B^2}}$$

$$W^* = \frac{A + \sqrt{A^2 - 4B^2}}{2}$$

# Analysis of the Model

## Interpretation

Our first conclusion:

- If  $A < 2B$  the only solution is  $U^* = 0$  e  $W^* = A$ .
- This represents a bare state. **A desert.**
- The condition  $A > 2B \Rightarrow a > \frac{2d\sqrt{bc}}{e}$  shows that there must be a minimum amount of precipitation to sustain vegetation.
- Moreover, the higher  $e$  the easier to have a state with vegetation. Recall that  $e$  represents the absorption rate. The higher, the better.
- On the other, the higher  $(b)$  and the death rate of the population,  $(d)$  easier it is to have a vegetationless solution.
- **Seems reasonable!**



# Analysis of the model

So, let  $A > 2B$

- If  $A > 2B$ , we can have two fixed points.
- What about their stability?.
- The linear stability analysis results in:
  - ▶ The fixed point  $U^* = 0$  and  $W^* = A$  is always **stable**, even if  $A \leq 2B$ .
  - ▶ The fixed point

$$U^* = \frac{2B}{A + \sqrt{A^2 - 4B^2}}, \quad W^* = \frac{A + \sqrt{A^2 - 4B^2}}{2}$$

is always **unstable**.

- ▶ The fixed point

$$U^* = \frac{2B}{A - \sqrt{A^2 - 4B^2}}, \quad W^* = \frac{A - \sqrt{A^2 - 4B^2}}{2}$$

is **stable** provided  $B > 2$ .

# Analysis of the Model

- So, if  $A > 2B$ , and  $B > 2$ , we have **two** stable fixed points, each of them with its basin of attraction.
- A pictorial view is as follows:

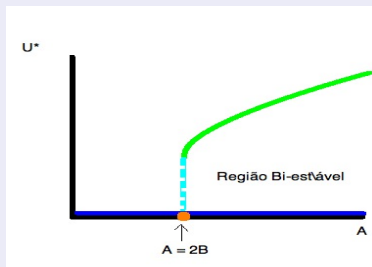


Figura :  $B$  is fixed, and we plot  $U^*$  ( the biomass ) in terms of  $A$ . The solution representing a desert ( $U^* = 0$ ) and the solution corresponding to vegetation cover are both stable

# Hysteresis

- The existence of a region of bi-stability ( $A > 2B$ ,  $B > 2$ ), can take us to the following situation.
- Take a fixed  $B$ . Consider that  $A$  can change slowly.
- Let us begin in the bi-stability. And let  $A$  decrease. At a certain moment,  $A$  will cross the critical value  $A = 2B$ .
- At this time a **sudden transition** occurs, a jump, in which  $U^* \rightarrow 0$ .  
**Desertification!!!**
- Suppose now that  $A$  begins again to increase - slowly. As  $U^* = 0$  is stable, even with  $A > 2B$  we will continue in the "desertic" region, as at the moment of crossing back the critical point we were in its the basin of attraction..
- **In summary:** if we begin with a certain  $A$ , decrease it  $A < 2B$  and then come back to our initial value of  $A$ , the state of the system can transit from  $U^* \neq 0$  to  $U^* = 0$ .
- This is called **Hysteresis**.

# Hysteresis

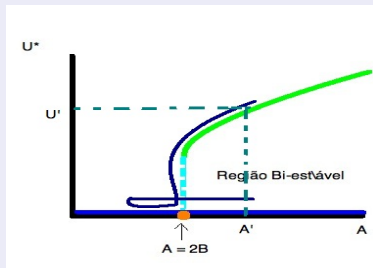


Figura :  $B$  is fixed.  $A$  begins at a value  $A'$  with  $U^* = U'$ , decreases, crosses a critical point at  $A = 2B$ . It goes to ,  $U^* \rightarrow 0$ . When we increase again  $A$ , even with  $A > 2B$ , we have  $U^* = 0$ .

Once the "desertic" state is attained it is not sufficient to change the external conditions back ( in our model, this is the rainfall) in order to get back a vegetation-cover state .

Terrible!

# Glory and Misery of the Model

## Glory

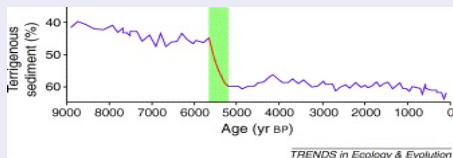


Figura : Estimated vegetation cover in the region of Sahara, over a long time span. We see a sudden change around 5500 BP.

- Existence of sudden transitions can be understood rather simply. The same kind of phenomenon appears in other systems as well.
- The model is **Simple**.

# Glory and Misery of the Model

## Misery



Figura : Desertification Region in China

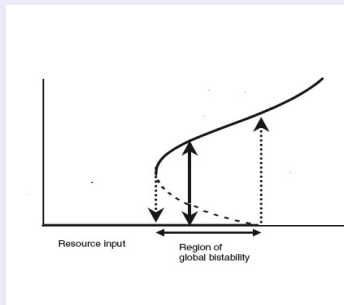


Figura : Senegal, at the Sahel region, south to Sahara.

- The model is **very simple**
- The transition is towards a completely vegetationless state. Actual desertification processes allow for remnants of vegetation.
- The model predicts an infinite bi-stability region... We could think that enough rain could reverse desertification.
- There are indeed better models.

## More realistic models

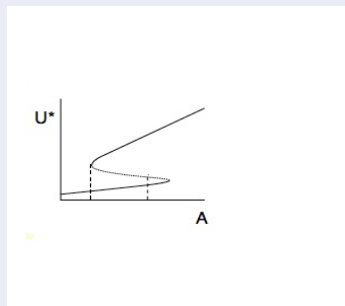
- More realistic models give bifurcation diagrams like the one below.



Biomass in terms of the rainfall, in a static case.. The blue curve represents **two transition regions**. The one to the left implies a **vegetation** → **desert** transition. To the right, a reversed transition.

## More realist models

- Still another curve.



This diagram is similar to the preceding one, but  $U^*$  does not tend to zero.



# Bibliography

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## Online Resources

- <http://www.ictp-saifr.org/mathbio3>
- <http://ecologia.ib.usp.br/ssmb/>

Thank you for your attention

